GEOMETRIC REPRESENTATION THEORY OF THE HILBERT SCHEMES PART I

ALEXANDER TSYMBALIUK

Abstract. We recall the classical action of the Heisenberg algebra $\mathcal{H}$ on the usual and equivariant homology of the Hilbert scheme of points on $\mathbb{C}^2$, due to Grojnowski-Nakajima.

1. Heisenberg algebra

In this section, we recall the definitions of the Heisenberg algebra and its Fock representation. These will be the key algebraic objects appearing in the forthcoming discussion.

Definition 1.1. The complex Lie algebra $\mathcal{H}$ with a basis $\{a_n, n \in \mathbb{Z}^*; h\}$ and a Lie bracket

$$[a_m, a_n] = m\delta^0_{m+n}h, \quad [a_m, h] = 0 \quad \text{for all } m, n \in \mathbb{Z}^*,$$

is called the *Heisenberg algebra* (here $\mathbb{Z}^* := \mathbb{Z}\backslash\{0\}$ and δ^j_i is the Kronecker delta function).

Let $\mathfrak{n}_+$ be the span of $\{a_m\}_{m>0}$, $\mathfrak{n}_-$ be the span of $\{a_m\}_{m<0}$, and $\mathfrak{h}$ be the span of h. Then $\mathcal{H} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Analogously to the case of simple Lie algebras, an $\mathcal{H}$-representation V is called the *highest weight representation of highest weight* $\lambda \in \mathbb{C}$ if there exists $v \in V$ such that

$$\mathfrak{n}_+(v) = 0, \quad h(v) = \lambda \cdot v, \quad U(\mathcal{H})(v) = V.$$

Our next result provides a classification for such $\mathcal{H}$-representations:

Proposition 1.1. *We have the following description of highest weight $\mathcal{H}$-representations:*
(a) Any highest weight representation of highest weight λ is a quotient of $\mathrm{Ind}^{\mathcal{H}}_{\mathfrak{n}_+\oplus\mathfrak{h}}\mathbb{C}_\lambda$*, where $\mathfrak{n}_+$ acts trivially on $\mathbb{C}_\lambda$, while h acts as a multiplication by λ.*
(b) The representation $\mathrm{Ind}^{\mathcal{H}}_{\mathfrak{n}_+\oplus\mathfrak{h}}\mathbb{C}_\lambda$ *can be realized as an $\mathcal{H}$-representation R^λ on the space $\mathbb{C}[x_1, x_2, \ldots]$ with generators acting in the following way:*

$$a_m \mapsto A_m = m\lambda\partial_{x_m}, \; a_{-m} \mapsto A_{-m} = x_m, \; h \mapsto H = \lambda\mathrm{Id}, \; m > 0.$$

(c) The representations R^λ are irreducible for $\lambda \neq 0$.

Exercise 1.2. *Prove Proposition 1.1.*

Definition 1.2. The representations R^λ are called the *Fock modules* over $\mathcal{H}$.

The Fock representation R^λ has a basis consisting of the elements

$$x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} = a_{-1}^{j_1} a_{-2}^{j_2} \cdots a_{-n}^{j_n}(1).$$

Define the degree of such a monomial as $\sum_{k=1}^n kj_k$. Let R^λ_j be the subspace of R^λ spanned by degree j monomials. Then $R^\lambda = \bigoplus_{j\geq 0} R^\lambda_j$ and $\dim(R^\lambda_j) = p(j)$–the number of partitions of j. Therefore the *q-dimension* of R^λ, defined by $\dim_q(R^\lambda) := \sum_{k\geq 0} \dim(R^\lambda_j)q^j$, is equal to:

$$\dim_q(R^\lambda) = \prod_{j=1}^{\infty} \frac{1}{1-q^j}. \tag{1}$$

2. Hilbert scheme of points

2.1. **The convolution machinery.**

Let us first recall the formalism of correspondences and convolutions in the general setting. For a locally compact topological space X, let $H_*(X)$ denote the usual homology groups, while $H_*^{BM}(X)$ denote the Borel-Moore homology of X (that is, homology with compact support).

Let M_1, M_2 be oriented smooth manifolds of dimensions d_1, d_2, and let $p_i : M_1 \times M_2 \to M_i$ be the natural projections. Suppose Z is an oriented submanifold $Z \subset M_1 \times M_2$ such that the projection $Z \to M_2$ is proper. Then the fundamental class $[Z] \in H_*^{BM}(M_1 \times M_2)$ defines a linear operator $[Z] : H_j^{BM}(M_1) \to H_{j+\dim Z - d_1}^{BM}(M_2)$, by $[Z](\gamma) = p_{2*}(p_1^*\gamma \cap [Z])$ for $\gamma \in H_*^{BM}(M_1)$. When viewed as an operator on (Borel-Moore) homology groups, we call $[Z]$ a *correspondence*. Using the Poincaré duality, we get the adjoint operator $[Z]^* : H_j(M_2) \to H_{j+\dim Z - d_2}(M_1)$, which can be formally defined by a similar formula $[Z]^*(\gamma) = p_{1*}(p_2^*\gamma \cap [Z])$ for $\gamma \in H_*(M_2)$.

Let $Z_1 \subset M_1 \times M_2,\ Z_2 \subset M_2 \times M_3$ be the subvarieties such that both projections $Z_1 \to M_2$ and $Z_2 \to M_3$ are proper. Then the operators $[Z_1] : H_*^{BM}(M_1) \to H_*^{BM}(M_2)$, $[Z_2] : H_*^{BM}(M_2) \to H_*^{BM}(M_3)$ are defined. Their composition $H_*^{BM}(M_1) \to H_*^{BM}(M_2) \to H_*^{BM}(M_3)$ is a correspondence given by the so called *convolution* class:

$$[Z_1] \star [Z_2] := p_{13*}(p_{12}^*[Z_1] \cap p_{23}^*[Z_2]) \in H_*^{BM}(Z_1 \circ Z_2),$$

where $Z_1 \circ Z_2 := p_{13}(p_{12}^{-1}(Z_1) \cap p_{23}^{-1}(Z_2)) \subset M_1 \times M_3$ (we use p_{ij} to denote the projection $M_1 \times M_2 \times M_3 \to M_i \times M_j$). Note that the projection $Z_1 \circ Z_2 \to M_3$ is proper and so $[Z_1] \star [Z_2]$ is well-defined. We also get the composition $[Z_1]^* \star [Z_2]^* : H_*(M_3) \to H_*(M_1)$ in a similar way.

Remark 2.1. All the above constructions also work for any $K \in H_*^{BM}(Z)$ instead of $[Z]$.

2.2. **Correspondences $Z_\alpha[i]$.**

Let X be a quasi-projective surface (our main example is $X = \mathbb{C}^2$), and $X^{[n]}$ the Hilbert scheme of n points in X. For $i > 0$, consider cycles $Z[i] \subset \bigsqcup_n X^{[n]} \times X^{[n+i]} \times X$ defined by

$$Z[i] = \bigsqcup Z^n[i],\ Z^n[i] := \left\{(J_1, J_2, x) \in X^{[n]} \times X^{[n+i]} \times X \mid J_1 \supset J_2,\ \mathrm{supp}(J_1/J_2) = \{x\}\right\}.$$

Let $\pi : Z[i] \to X$ be the projection to the last factor.

We also define $Z^n[-i] \subset X^{[n]} \times X^{[n-i]} \times X$ $(n \geq i)$ and $\pi : Z^n[-i] \to X$ in a similar way.

Exercise 2.1. *The dimension of* $Z^n[i]$ *is given by* $\dim_{\mathbb{C}}(Z^n[i]) = 2n + i + 1$.

Remark 2.2. In all such dimension counting arguments we need to know $\dim_{\mathbb{C}} s^{-1}(n[x]) = n-1$ for any point $x \in \mathbb{C}^2$, where $s : (\mathbb{C}^2)^{[n]} \to \mathrm{Sym}^n(\mathbb{C}^2)$ is the Hilbert-Chow map.

Consider the homology classes $\alpha \in H_*^{BM}(X), \beta \in H_*(X)$ and let $p_{12} : Z[\pm i] \to X^{[n\mp i]} \times X^{[n]}$ be the projection to the product of the first two factors. We define

$$Z_\alpha[i] := p_{12*}(\pi^*\alpha \cap [Z[i]]),\quad Z_\beta[-i] := p_{12*}(\pi^*\beta \cap [Z[-i]]),\ i > 0.$$

These should viewed as $Z_\alpha[i], Z_\beta[-i] \in \prod_n H_*^{BM}(X^{[n\mp i]} \times X^{[n]})$.

Remark 2.3. The projection p_{12} is proper, so p_{12*} is well-defined.

Applying the machinery of Section 2.1 to the cycles $Z_\alpha[i], Z_\beta[-i]$, we get the correspondences $H_*(X^{[n]}) \to H_*(X^{[n\mp i]})$. Our next remark provides more details on this construction.

Remark 2.4. The projections $p_{13} : Z^n[i] \to X^{[n]} \times X$ and $p_2 : Z^n[i] \to X^{[n+i]}$ are proper. They induce the correspondences $H_*(X^{[n]} \times X) \xrightarrow{\phi} H_*(X^{[n+i]})$, $H_*(X^{[n+i]}) \xrightarrow{\psi} H_*(X^{[n]} \times X)$. Then $Z_\beta[-i](u) = \phi(u \otimes \beta)$, $Z_\alpha[i](v) = \langle \psi(v), 1 \otimes \alpha \rangle$ for $u \in H_*(X^{[n]}), v \in H_*(X^{[n+i]})$. This argument also clarifies why α and β are chosen from the B-M homology or the homology groups.

According to Exercise 2.1, we have $Z_\alpha[i] : H_{2n+k}(X^{[n]}) \to H_{2(n-i)+k+\deg \alpha - 2}(X^{[n-i]})$.

2.3. **Main result.**

Let M be the direct sum of the homology groups $M := \bigoplus_{n\geq 0} H_*(X^{[n]})$. Then we get the operators $Z_\alpha[i], Z_\beta[j] \in \mathrm{End}(M)$.

Theorem 2.2. [N, Theorem 8.13] *The following relation holds:*

$$Z_\alpha[i]Z_\beta[j] - (-1)^{\deg\alpha\deg\beta} Z_\beta[j]Z_\alpha[i] = (-1)^{i-1} i\delta^0_{i+j}\langle\alpha,\beta\rangle\,\mathrm{Id}_M,$$

where $\langle\alpha,\beta\rangle \in \mathbb{C}$ *is defined by* $\langle\alpha,\beta\rangle := p_{X*}(\alpha\cap\beta)$ *with* $p_X : X \to \mathrm{pt}$.

Corollary 2.3. *If* $\deg\alpha\deg\beta$ *is even then the operators* $\mathfrak{q}_\beta[-i] := Z_\beta[-i],\ \mathfrak{q}_\alpha[i] := (-1)^{i-1}Z_\alpha[i]$ *define an* $\mathcal{H}$*-action on* M *of central charge* $\langle\alpha,\beta\rangle$.

For $X = \mathbb{C}^2$, up to proportionality there is only one nontrivial choice of such α and β: $\alpha = [\mathbb{C}^2], \beta = [\mathrm{pt}]$. Let $M' \subset M$ be a submodule $M' = \mathcal{H}(\mathbf{1})$, where $H_0(X^{[0]}) \simeq \mathbb{C}\cdot\mathbf{1}$. Since $Z_\alpha[i](\mathbf{1}) = 0$ for $i > 0$, M' is isomorphic to the Fock module over $\mathcal{H}$.

According to [N, Section 5], the q-dimension of M equals[1] $\sum_{n\geq 0} q^n \dim H_\bullet(X^{[n]}) = \prod_{j\geq 1}\frac{1}{1-q^j}$. An easy way to see this is to use a contractable action of the one-dimensional subtorus $T_1 \subset T$ on $X^{[n]}$ (e.g. $T_1 = \{t^N, t^{N+1}\}$ for $N > n$). This yields a cell-decomposition of $X^{[n]}$ with the number of cells equal to the number of fixed points, i.e., the number of size n Young diagrams.

Comparing this to the formula (1), we get

Theorem 2.4. *The representation* M *is the Fock module over* $\mathcal{H}$.

Remark 2.5. For general X, one incorporates all choices of α, β into an action of the Heisenberg superalgebra $\mathcal{A}(V)$, corresponding to the super vector space $V = H_{\mathrm{even}}(X) \oplus H_{\mathrm{odd}}(X)$. Same argument proves that M is the Fock module over $\mathcal{A}(V)$ (see Appendix A).

2.4. **Baby example of Theorem 2.2.**

Let us consider the first nontrivial example: $i = 1,\ j = -1$ for $\alpha = [X],\ \beta = [x_0]$. We verify $Z_\alpha[1]Z_\beta[-1] - Z_\beta[-1]Z_\alpha[1] = \mathrm{Id}$ when viewed as operators on $H_0(X^{[n]})$. Since $H_0(X^{[n]})$ is one dimensional, it suffices to check the above for the fundamental class of any $J_0 \in X^{[n]}$. A generic J_0 can be identified with $J_0 = \{x_1, \ldots, x_n\}$ for n pairwise distinct points $x_1, \ldots, x_n \in X\backslash\{x_0\}$.

When applying $Z_\beta[-1]$ to J_0 we just get an ideal corresponding to $\{x_0, x_1, \ldots, x_n\}$. Next, the correspondence $Z_\alpha[1]$ deletes one of the points so that

$$Z_\alpha[1]Z_\beta[-1](J_0) = \{x_1, \ldots, x_n\} + \sum_{i=1}^n \{x_0, x_1, \ldots, \widehat{x_i}, \ldots, x_n\},$$

where $\widehat{x_i}$ means that x_i is missing. Analogously we get:

$$Z_\beta[-1]Z_\alpha[1](J_0) = \sum_{i=1}^n \{x_0, x_1, \ldots, \widehat{x_i}, \ldots, x_n\}.$$

Therefore, we indeed have $(Z_\alpha[1]Z_\beta[-1] - Z_\beta[-1]Z_\alpha[1])(J_0) = J_0$.

2.5. **Sketch of the proof of Theorem 2.2.**

We outline only the main ingredients in the proof of Theorem 2.2. There are three cases to be considered: $i, j > 0,\ 0 > i, j$, and $i > 0 > j$. We will discuss $i, j > 0$ now (the case $i, j < 0$ is analogous), while $i > 0 > j$ is considered in Appendix B.

For $i, j > 0$, the composition of correspondences $Z^{n-i-j}_\alpha[i]Z^{n-j}_\beta[j]$ is given by the convolution

$$Z^{n-i-j}_\alpha[i] \star Z^{n-j}_\beta[j] = p_{13*}(p^*_{124}[Z^{n-i-j}[i]] \cap \pi^*_1(\alpha) \cap p^*_{235}[Z^{n-j}[j]] \cap \pi^*_2(\beta)),$$

[1] For a general X, we have $\sum_{n\geq 0} q^n \dim H_\bullet(X^{[n]}) = \prod_{j\geq 1}\frac{(1+q^j)^{\dim H_{\mathrm{odd}}(X)}}{(1-q^j)^{\dim H_{\mathrm{even}}(X)}}$ ([GS] and [N, Section 6]).

where $p_{124}, p_{235}, p_{13}, \pi_1 := p_4, \pi_2 := p_5$ are the projections of $X^{[n-i-j]} \times X^{[n-j]} \times X^{[n]} \times X \times X$ to the corresponding (product of) factors.

This class is set-theoretically supported at $Z^{n-i-j}[i] \circ Z^{n-j}[j]$ which is equal to

$$Z[i,j;n] = \{(J_1, J_2, J_3, x, y) \mid J_1 \supset J_2 \supset J_3,\ \mathrm{supp}(J_1/J_2) = \{x\},\ \mathrm{supp}(J_2/J_3) = \{y\}\}.$$

Consider the stratification $Z[i,j;n] = Z[i,j;n]_1 \sqcup Z[i,j;n]_2$ corresponding to $x = y$ and $x \neq y$.

Remark 2.6. It turns out that $Z[i,j;n]_2$ is a *good* stratum (see Example 2.5), while $Z[i,j;n]_1$ is a *neglectable* stratum, since it is of smaller dimension.

To be more precise, we have the following result:

Exercise 2.5. *(a)* $p_{124}^{-1}(Z^{n-i-j}[i])$ *and* $p_{235}^{-1}(Z^{n-j}[j])$ *intersect transversely along* $Z[i,j;n]_2$.
(b) $\dim_{\mathbb{C}} Z[i,j;n]_2 = \dim_{\mathbb{C}} Z[i,j;n] = 2n - i - j + 2$.
(c) $p_{124}^*[Z[i]] \cap p_{235}^*[Z[j]]$ *is a degree* $4n - 2i - 2j + 4$ *B-M homology class of* $Z[i,j;n]$.

Let $\bar{Z}[i,j] := \overline{Z[i,j]_2}$. According to the above exercise, we have:

$$p_{124}^*[Z[i]] \cap p_{235}^*[Z[j]] = [\bar{Z}[i,j]] + \iota_{ij*}(\gamma) \text{ for some } \gamma \in H_*^{BM}(Z[i,j]_1),$$
$$p_{124}^*[Z[j]] \cap p_{235}^*[Z[i]] = [\bar{Z}[j,i]] + \iota_{ji*}(\gamma') \text{ for some } \gamma' \in H_*^{BM}(Z[j,i]_1),$$

where $\iota_{ij} : Z[i,j;n]_1 \hookrightarrow Z[i,j;n],\ \iota_{ji} : Z[j,i;n]_1 \hookrightarrow Z[j,i;n]$ are the strata inclusions.

It is easy to see that there exists an isomorphism $Z[i,j;n]_2 \xrightarrow{\sim} Z[j,i;n]_2$ interchanging π_1, π_2. In other words, given an ideal J_2 such that $\mathrm{supp}(J_1/J_2) = \{x\} \neq \{y\} = \mathrm{supp}(J_2/J_3)$ there exists a unique ideal J_2' such that $J_1 \supset J_2' \supset J_3$ and $\mathrm{supp}(J_1/J_2') = \{y\},\ \mathrm{supp}(J_2'/J_3) = \{x\}$.

It turns out that the remaining summands do not give any contribution:

Lemma 2.6. *We have* $p_{13*}(\pi_1^*(\alpha) \cap \pi_2^*(\beta) \cap \iota_{ij*}(\gamma)) = 0$.

Combining this lemma with the above argument, we get:

$$Z_\beta[j] Z_\alpha[i] = (-1)^{\deg\alpha \deg\beta} Z_\alpha[i] Z_\beta[j].$$

Proof of Lemma 2.6.
Consider $\pi : Z[i,j]_1 \to X$ defined by $\pi = \pi_1 = \pi_2$. By the projection formula, we have:

$$\pi_1^*(\alpha) \cap \pi_2^*(\beta) \cap \iota_{ij*}(\gamma) = \iota_{ij*}(\pi^*(\alpha \cap \beta) \cap \gamma).$$

To calculate the push-forward of this class along p_{13}, let us note that it factors through

$$Z[i,j;n]_1 \xrightarrow{\phi_{ij}} Z^{n-i-j}[i+j] \xrightarrow{p_{12}} X^{[n-i-j]} \times X^{[n]}.$$

Applying the projection formula again, we get:

$$p_{13*}\iota_{ij*}(\pi^*(\alpha \cap \beta) \cap \gamma) = p_{12*}(\pi^*(\alpha \cap \beta) \cap \phi_{ij*}\iota_{ij*}(\gamma)),$$

where π on the right hand side denotes the projection $Z^{n-i-j}[i+j] \to X$.

It remains to note that $\dim_{\mathbb{C}}(Z^{n-i-j}[i+j]) = 2n - i - j + 1$, while $\phi_{ij*}\iota_{ij*}(\gamma)$ is a degree $2(2n-i-j+2)$ homology class of $Z^{n-i-j}[i+j]$. Hence $\phi_{ij*}\iota_{ij*}(\gamma) = 0$ and the result follows. □

In Appendix B we apply the same arguments to the case $i > 0 > j$:
∘ The summands corresponding to the stratum $Z[i,j;n]_2$ can be handed in the same way.
∘ To deal with the summands coming from the homology classes of $Z[i,j;n]_1$, we introduce an auxiliary subvariety $L \subset X^{[n-i-j]} \times X^{[n]} \times X$ playing the role of $Z^{n-i-j}[i+j]$ from the above proof. If $j \neq -i$, the same dimension counting arguments apply to show that the corresponding summands are zero. If $j = -i$, then the only irreducible component of L of the desired dimension is $\triangle_{X^{[n]}} \times X$. Therefore, the corresponding summand acts as a multiplication by a constant.

3. Equivariant setting for $X = \mathbb{C}^2$

In this section we consider the simplest surface $X = \mathbb{C}^2$, but in the equivariant setting.

3.1. **General results on equivariant (co)homology.**

Let $T := (\mathbb{C}^*)^r$ be the r-dimensional torus, $V_N := (\mathbb{C}^{N+1})^r$–natural T-module with each $\mathbb{C}^*$ acting on the corresponding copy of $\mathbb{C}^{N+1}$ by multiplication, and M a T-variety.

Let us recall the key results on equivariant (co)homology (see the seminar notes [M]):

• Define $H^i_T(M) := \varprojlim H^i(M_N)$, where $M_N := (V_N\backslash\{0\}) \times_T M$.[2]

∘ For $M = \mathrm{pt}$, we get $H^*_T(\mathrm{pt}) \simeq \mathbb{C}[a_1, \ldots, a_r]$, where $a_i := \varprojlim c_1(\mathcal{O}(1)_i)$. Actually, we have a canonical identification $H^*_T(\mathrm{pt}) \simeq \mathbb{C}[\mathrm{Lie}\, T]$ (that is independent of the isomorphism $T \xrightarrow{\sim} (\mathbb{C}^*)^r$).

∘ There is a cup product $H^i_T(M) \otimes H^j_T(M) \to H^{i+j}_T(M)$.

∘ For any T-equivariant map $f : M_1 \to M_2$, there is a pull-back map $H^*_T(M_2) \to H^*_T(M_1)$.

In particular, $H^*_T(M)$ is an algebra over $H^*_T(\mathrm{pt}) \simeq \mathbb{C}[a_1, \ldots, a_r]$ (by considering $M \to \mathrm{pt}$).

∘ For any subtorus $T' \subset T$ we have a pull-back homomorphism $H^*_T(M) \to H^*_{T'}(M)$, induced by the T/T'-fibration $M \times_{T'} ET \to M \times_T ET$.

In particular, for $T' = \{1\} \subset T$ we get a natural homomorphism $H^*_T(M) \to H^*(M)$.

∘ For a T-equivariant bundle E over M, define an equivariant Chern class $c_i(E) := \varprojlim c_i(E_N)$.

For $M = \mathrm{pt}$, a T-equivariant vector bundle E over M is just a representation of T. Then $c_i(E)$ is the i-th elementary symmetric function of weights of E, viewed as functionals $\mathrm{Lie}\, T \to \mathbb{C}$.

• Define $H^{T,BM}_i(M) := \varprojlim H^{BM}_{i+4N-4r}(M_N)$ (the degree shift agrees with the Poincaré duality).

∘ The cap product on $H^{BM}_*(M_N)$ induces an $H^*_T(M)$-module structure on $H^{T,BM}_*(M)$.

∘ For a smooth M, we define the equivariant fundamental class $[M] \in H^{T,BM}_{\dim_{\mathbb{R}} M}(M)$ as $\varprojlim [M_N]$.

∘ The Poincaré duality on M_N induces the isomorphism $H^i_T(M) \simeq H^{T,BM}_{m-i}(M)$.

∘ For a proper $f : M_1 \to M_2$, there is a push-forward map $f_* : H^{T,BM}_*(M_1) \to H^{T,BM}_*(M_2)$.

∘ Similarly to the cohomological case, we have a natural homomorphism $H^{T,BM}_i(M) \to H^{BM}_i(M)$.

• Finally, we define equivariant homology groups $H^T_i(M)$ as $H^T_i(M) := H^i_T(M)^*$.

∘ For a smooth M, we have the cap product $\cap : H^{T,BM}_i(M) \otimes H^T_j(M) \to H^T_{i+j-\dim_{\mathbb{R}} M}(M)$.

∘ Composing $\cap$ with push-forward to a point and using the Poincaré duality, we obtain the intersection pairing $\langle\cdot,\cdot\rangle : H^{T,BM}_i(M) \otimes H^T_j(M) \longrightarrow H^T_{i+j-\dim_{\mathbb{R}} M}(\mathrm{pt}) \simeq H^{\dim_{\mathbb{R}} M-i-j}_T(\mathrm{pt})$.

• Let us define $H^T_*(M)_{\mathrm{loc}} := H^T_*(M) \otimes_{H^*_T(\mathrm{pt})} \mathrm{Frac}(H^*_T(\mathrm{pt}))$. The inclusion $\iota : M^T \hookrightarrow M$ induces

$$\iota_* : H^T_*(M^T)_{\mathrm{loc}} \xrightarrow{\sim} H^T_*(M)_{\mathrm{loc}}.$$

This result is known as the *localization theorem.*

Consider the decomposition $M^T = \bigsqcup M_\alpha$ into connected components. Let $\iota_\alpha : M_\alpha \hookrightarrow M$ be the natural inclusion, while N_α denotes its normal bundle. If M and M_α are both smooth, then Poincaré duality induces a map $\iota_{\alpha*} : H^T_*(M_\alpha) \to H^T_*(M)$. According to the Thom isomorphism, the composition $\iota^*_\alpha \iota_{\alpha*}$ is given by the cap product with an equivariant Euler class $e(N_\alpha)$. The operator $e(N_\alpha) \cap \bullet$ is invertible in $H^T_*(M_\alpha)_{\mathrm{loc}}$; let $\frac{1}{e(N_\alpha)}$ denote its inverse. The following result is known as the *Atiyah-Bott-Berline-Vergne fixed point formula*:

$$p_{M*}(\omega) = \sum_\alpha p_{\alpha*}\left(\frac{1}{e(N_\alpha)}\iota^*_\alpha \omega\right),\ \omega \in H^T_*(M),$$

where p_M, p_α are the projections of M, M_α onto a point (assuming M is proper).

[2] This definition is consistent with the usual one, since the universal T-bundle $ET \to BT$ is just the limit of the T-bundles $V_N\backslash\{0\} \to (\mathbb{P}^N)^r$. Moreover, the sequence $\{H^i(M_N)\}$ stabilizes as $N \to \infty$.

3.2. **Main result in equivariant setting.**

Consider an action of the two dimensional torus T on X given by $(t_1,t_2)\cdot(x,y)=(t_1x,t_2y)$, where x,y are the standard coordinates on $\mathbb{C}^2$. We have induced T-actions on $X^{[n]}$ and S^nX, so that the Hilbert-Chow morphism $s: X^{[n]}\to \mathrm{Sym}^n X$ is T-equivariant.

Constructions of the previous section provide correspondences

$$Z_\alpha[i]: H_*^{T,BM}(X^{[n]})\to H_*^{T,BM}(X^{[n-i]}),\ Z_\beta[j]: H_*^{T,BM}(X^{[n-i]})\to H_*^{T,BM}(X^{[n]})$$

for any $\alpha\in H_*^T(X),\ \beta\in H_*^{T,BM}(X)$.

Remark 3.1. One can alternatively work with H_*^T, but then we should swap the homology groups: take $\alpha\in H_*^{T,BM}(X),\ \beta\in H_*^T(X)$ (see Remark 2.4).

Let M^T be the direct sum of the equivariant Borel-Moore homology groups:

$$M^T:=\bigoplus_{n\geq 0} H_*^{T,BM}(X^{[n]}).$$

The following result is a T-equivariant analogue of Theorem 2.2:

Theorem 3.1. *For any $\alpha\in H_*^T(X),\ \beta\in H_*^{T,BM}(X)$, we have*

$$[Z_\alpha[i],Z_\beta[j]]=(-1)^{i-1}i\delta_{i+j}^0\langle\alpha,\beta\rangle\mathrm{Id}_{M^T}.$$

This theorem is proved in the same way as its non-equivariant analogue. One of the key arguments on vanishing of some cycles remains the same, since $H_i^{T,BM}(M)=0$ for $i>\dim_{\mathbb{R}} M$. In other words, one can show that $[Z_\alpha[i],Z_\beta[j]]=c_{i,n}^T\langle\alpha,\beta\rangle\,\mathrm{Id}$ for some $c_{i,n}^T\in H_T^*(\mathrm{pt})$. Taking the non-equivariant limit, we find that $c_{i,n}^T$ is given by the same formula as in the non-equivariant setting: $c_{i,n}^T=c_{i,n}=(-1)^{i-1}i$.[3]

3.3. **Young diagrams.**

In this section we introduce some common notation for Young diagrams:

- For a Young diagram λ we use λ^* to denote its conjugate.
- For a Young diagram λ we write $\lambda=(1^{n_1}2^{n_2}\cdots)$, where $n_i=\#\{k:\lambda_k=i\}$.
- For a Young diagram λ its length $l(\lambda)$ is defined by $l(\lambda)=\max\{k:\lambda_k\neq 0\}=\lambda_1^*$.
- We say that $\lambda\geq\mu$ iff $|\lambda|=|\mu|$ and $\lambda_1+\cdots+\lambda_i\geq\mu_1+\cdots+\mu_i$ for all i.
- We say that $\lambda\succeq\mu$ iff $|\lambda|=|\mu|$ and there exists i such that

$$\lambda_1+\cdots+\lambda_i>\mu_1+\cdots+\mu_i,\ \text{while}\ \lambda_1+\cdots+\lambda_j=\mu_1+\cdots+\mu_j\ \text{for}\ j<i.$$

- For a box $\square\in\lambda$ with coordinates (i,j) we define $l(\square):=\lambda_j^*-i,\ a(\square):=\lambda_i-j$.

3.4. **Fixed points of $X^{[n]}$.**

Let us recall the bijection between the T-fixed points of $X^{[n]}$ and Young diagrams of size n. For a Young diagram $\lambda\vdash n$, the corresponding T-fixed point $\xi_\lambda\in X^{[n]}$ is defined by the ideal

$$J_\lambda:=(y^{\lambda_1},xy^{\lambda_2},x^2y^{\lambda_3},\ldots,x^{\lambda_1^*}).$$

Note that the quotient $Q_\lambda:=\mathbb{C}[x,y]/J_\lambda$ has a basis consisting of the images of monomials

$$\{x^{i-1}y^{j-1}|1\leq i\leq l(\lambda),1\leq j\leq\lambda_i\}.$$

The following formula for the T-character of the tangent space to $X^{[n]}$ at the fixed point $\{\xi_\lambda\}$ will be important for us (more details in our next class):

$$\mathrm{ch}\ T_{\xi_\lambda}(X^{[n]})=\sum_{\square\in\lambda}\left(t_1^{l(\square)+1}t_2^{-a(\square)}+t_1^{-l(\square)}t_2^{a(\square)+1}\right). \tag{2}$$

[3] We use the compatibility of the natural maps $H_*^{T,BM}(\bullet)\to H_*^{BM}(\bullet)$ with push-forwards and pull-backs.

Appendix A. Heisenberg superalgebras

A.1. **Clifford algebra.**

Definition A.1. The associative algebra $\mathcal{C}$ generated by $\{a_n, n \in \mathbb{Z}^*; h\}$ with defining relations
$$\{a_m, a_n\} := a_m a_n + a_n a_m = |m|\delta^0_{m+n} h, \quad [a_m, h] = 0 \quad \text{for all } m, n \in \mathbb{Z}^*,$$
is called the *Clifford algebra.*

Let us consider subalgebras $\mathcal{C}_-, \mathcal{C}_0, \mathcal{C}_+$ of $\mathcal{C}$ defined by $\mathcal{C}_\pm := \mathbb{C}[a_{\pm1}, a_{\pm2}, \ldots]$ and $\mathcal{C}_0 := \mathbb{C}[h]$. The natural multiplication map $m : \mathcal{C}_- \otimes \mathcal{C}_0 \otimes \mathcal{C}_+ \to \mathcal{C}$ is an isomorphism.

A $\mathcal{C}$-representation V is called the *highest weight representation of highest weight* $\lambda \in \mathbb{C}$ if
$$\exists\, v \in V \quad \text{such that} \quad \mathcal{C}_+(v) = 0, \quad h(v) = \lambda \cdot v, \quad \mathcal{C}(v) = V.$$

It is clear that any highest weight $\mathcal{C}$-representation is again a quotient of $\mathrm{Ind}^{\mathcal{C}}_{\mathcal{C}_{\geq 0}} \mathbb{C}_\lambda$, where $\mathcal{C}_{\geq 0} \subset \mathcal{C}$ is the image of $\mathcal{C}_0 \otimes \mathcal{C}_+$ under m. Moreover, the representation $\mathrm{Ind}^{\mathcal{C}}_{\mathcal{C}_{\geq 0}} \mathbb{C}_\lambda$ can be now realized on the full exterior algebra $F^\lambda := \wedge^\bullet \langle dx_1, dx_2, \ldots \rangle$ by the following operators:
$$a_m \mapsto A_m = m\lambda \frac{\partial}{\partial x_m} \lrcorner, \; a_{-m} \mapsto A_{-m} = dx_m \wedge, \; h \mapsto H = \lambda \mathrm{Id}, \; m > 0.$$
Notice that F^λ has a basis consisting of the elements
$$dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_n} = a_{-j_1} a_{-j_2} \cdots a_{-j_n}(1) \quad (j_1 > j_2 > \cdots > j_n).$$

Define the degree of such a monomial as $\sum j_k$. Let F^λ_j be the subspace of F^λ spanned by degree j monomials. Then $F^\lambda = \bigoplus_{j \geq 0} F^\lambda_j$ and the q-dimension of F^λ is given by
$$\dim_q(F^\lambda) = \prod_{j \geq 1} (1 + q^j). \tag{3}$$

A.2. **Heisenberg superalgebra.**

The purpose of this section is to unify the parallel constructions for $\mathcal{H}$ and $\mathcal{C}$.

Consider a finite dimensional super vector space $V = V_0 \oplus V_1$ with a non-degenerate bilinear form $(\cdot, \cdot)$ satisfying $(u, v) = (-1)^{\deg u \deg v}(v, u)$ for homogeneous elements $u, v \in V$. Define a super vector space $\widetilde{V} := V \otimes t\mathbb{C}[t] \oplus V \otimes t^{-1}\mathbb{C}[t^{-1}]$ with a bilinear form $(u \otimes t^i, v \otimes t^j) := i\delta^0_{i+j}(u, v)$.

Definition A.2. The free Lie algebra on $\widetilde{V} \oplus \mathbb{C} \cdot h$ with the defining relations
$$[\widetilde{u}, \widetilde{v}] = (\widetilde{u}, \widetilde{v})h, \; [\widetilde{u}, h] = 0, \; \widetilde{u}, \widetilde{v} \in \widetilde{V},$$
is called the *Heisenberg superalgebra*, and will be denoted by $\mathcal{A}(V)$.

Analogously to $\mathcal{H}$, we define subalgebras $\mathcal{A}(V)_-, \mathcal{A}(V)_0, \mathcal{A}(V)_+$ of $\mathcal{A}(V)$ as those spanned by $V \otimes t^{-1}\mathbb{C}[t^{-1}]$, h, and $V \otimes t\mathbb{C}[t]$, respectively. One can analogously introduce the notion of the highest weight $\mathcal{A}(V)$-representation of highest weight $\lambda \in \mathbb{C}$.

If $\lambda \neq 0$, $R^\lambda_V := \mathrm{Ind}^{\mathcal{A}(V)}_{\mathcal{A}(V)_{\geq 0}} \mathbb{C}_\lambda$ is the only such $\mathcal{A}(V)$-representation. This representation is called the *Fock module* over the Heisenberg superalgebra $\mathcal{A}(V)$ and can be naturally realized on the super-symmetric algebra $S^*(V \otimes t^{-1}\mathbb{C}[t^{-1}]) \simeq S^\bullet(V_0 \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes \Lambda^\bullet(V_1 \otimes t^{-1}\mathbb{C}[t^{-1}])$. This vector space is graded by $\deg(V \otimes t^{-n}) = n$. Combining formulas (1) and (3), we get
$$\dim_q(R^\lambda_V) = \prod_{j \geq 1} \frac{(1 + q^j)^{\dim V_1}}{(1 - q^j)^{\dim V_0}}. \tag{4}$$

Remark A.1. According to Theorem 2.2, we get an action of the Heisenberg superalgebra $\mathcal{A}(V)$ on $M = \bigoplus_n H_*(X^{[n]})$, where the super vector space V is given by $V = H_{\mathrm{even}}(X) \oplus H_{\mathrm{odd}}(X)$.

Appendix B. Proof of Theorem 2.2 for $i > 0 > j$

In this case we can similarly define $Z[i,j;n] \subset X^{[n-i-j]} \times X^{[n-j]} \times X^{[n]} \times X \times X$ by

$$Z[i,j;n] = \{(J_1, J_2, J_3, x, y) \mid J_1 \supset J_2 \subset J_3,\ \mathrm{supp}(J_1/J_2) = \{x\},\ \mathrm{supp}(J_3/J_2) = \{y\}\}.$$

This subvariety has a natural stratification $Z[i,j;n] = Z[i,j;n]_1 \sqcup Z[i,j;n]_2$ as before. Let $\iota_{ij} : Z[i,j;n]_1 \hookrightarrow Z[i,j;n]$ be the strata inclusion, while $\bar{Z}[i,j] := \overline{Z[i,j;n]_2}$. We have

$$Z_\alpha[i]Z_\beta[j] = p_{13*}([\bar{Z}[i,j]] \cap \pi_1^*(\alpha) \cap \pi_2^*(\beta)) + p_{13*}(\iota_{ij*}(\gamma_1) \cap \pi_1^*(\alpha) \cap \pi_2^*(\beta))$$

$$Z_\beta[j]Z_\alpha[i] = p_{13*}([\bar{Z}[j,i]] \cap \pi_1^*(\beta) \cap \pi_2^*(\alpha)) + p_{13*}(\iota_{ji*}(\gamma_2) \cap \pi_1^*(\beta) \cap \pi_2^*(\alpha))$$

for some $\gamma_1 \in H_*^{BM}(Z[i,j;n]_1), \gamma_2 \in H_*^{BM}(Z[j,i;n]_1)$. Analogously to the case $i,j > 0$:

$$p_{13*}([\bar{Z}[j,i]] \cap \pi_1^*(\beta) \cap \pi_2^*(\alpha)) = (-1)^{\deg\alpha \deg\beta} p_{13*}([\bar{Z}[i,j]] \cap \pi_1^*(\alpha) \cap \pi_2^*(\beta)).$$

Let us now compute the classes $p_{13*}(\iota_{ij*}(\gamma_1) \cap \pi_1^*(\alpha) \cap \pi_2^*(\beta))$ and $p_{13*}(\iota_{ji*}(\gamma_2) \cap \pi_1^*(\beta) \cap \pi_2^*(\alpha))$.

Consider $\pi : Z[i,j;n]_1 \to X$ defined by $\pi = \pi_1 = \pi_2$. By the projection formula, we have:

$$\iota_{ij*}(\gamma_1) \cap \pi_1^*(\alpha) \cap \pi_2^*(\beta) = \iota_{ij*}(\pi^*(\alpha \cap \beta) \cap \gamma_1).$$

Next, we analyze $p_{134}(Z[i,j;n]_1)$. Let us first introduce $L \subset X^{[m]} \times X^{[n]} \times X$ by

$$L = \left\{ (J_1, J_3, x) \subset X^{[m]} \times X^{[n]} \times X \mid J_1 = J_3 \text{ outside } x,\ s(J_1) = s(J_3) + (m-n)[x] \right\},$$

where s denotes the Hilbert-Chow map as before. The following result is straightforward:

Lemma B.1. [N, Lemma 8.32] *Decomposition of L into irreducible components is as follows:*
(a) If $m > n$, then L has one irreducible component of complex dimension $m + n + 1$, while all other components have smaller dimension.
(b) If $m = n$, then L has one irreducible component $L_0 = \triangle_{X^{[n]}} \times X$ of complex dimension $2n+2$, n irreducible components $L_1, \ldots, L_n$ of complex dimension $2n$, while all other components have smaller dimension.

Note that $p_{134}(Z[i,j;n]_1)$ is contained in L with $m = n - i - j$. We have three cases:
$\circ$ $i + j < 0$. Let $f : L \to X$ be the projection to the last factor. Note that the composition $p_{13} : Z[i,j;n]_1 \to X^{[m]} \times X^{[n]}$ factors through $p_{13} : L \to X^{[m]} \times X^{[n]}$. Therefore

$$p_{13*}(\iota_{ij*}(\gamma_1) \cap \pi_1^*(\alpha) \cap \pi_2^*(\beta)) = p_{13*}(p_{134*}\iota_{ij*}(\gamma_1) \cap f^*(\alpha \cap \beta)).$$

Applying Lemma B.1, we get $\dim_{\mathbb{C}} L = 2n - i - j + 1$ which is less then the expected dimension.[4]

Hence $p_{13*}(\iota_{ij*}(\gamma_1) \cap \pi_1^*(\alpha) \cap \pi_2^*(\beta)) = 0$. Analogously $p_{13*}(\iota_{ji*}(\gamma_2) \cap \pi_1^*(\beta) \cap \pi_2^*(\alpha)) = 0$.
$\circ$ $i + j > 0$. This case is reduced to the previous by interchanging J_1 and J_3.
$\circ$ $i + j = 0$. This is the only case when the contribution might be nonzero.

Let $j : \dot{L} := L \backslash L_0 \hookrightarrow L$ be the natural inclusion. Analogous arguments establish

$$j^* p_{134*}(\iota_{ij*}(\gamma_1) \cap \pi_1^*(\alpha) \cap \pi_2^*(\beta)) = 0,\ j^* p_{134*}(\iota_{ji*}(\gamma_2) \cap \pi_1^*(\beta) \cap \pi_2^*(\alpha)) = 0.$$

Both $p_{134*}\iota_{ij*}(\gamma_1)$ and $p_{134*}\iota_{ji*}(\gamma_2)$ are degree $2(2n+2) = \dim_{\mathbb{R}} L_0$ homology classes and so

$$p_{134*}\iota_{ij*}(\gamma_1) - (-1)^{\deg\alpha \deg\beta} p_{134*}\iota_{ji*}(\gamma_2) = c_{i,n}[L_0] \text{ for some } c_{i,n} \in \mathbb{C}.$$

Therefore:

$$p_{12*}((p_{134*}\iota_{ij*}(\gamma_1) - (-1)^{\deg\alpha \deg\beta} p_{134*}\iota_{ji*}(\gamma_2)) \cap \pi^*(\alpha \cap \beta)) = c_{i,n}[\triangle_{X^{[n]}}] \cdot \langle \alpha, \beta \rangle.$$

Hence $Z_\alpha[i]Z_\beta[-i] - (-1)^{\deg\alpha \deg\beta} Z_\beta[-i]Z_\alpha[i] = c_{i,n}\langle \alpha, \beta \rangle \,\mathrm{Id}_M$. It was first proved in [ES] that $c_{i,n} = (-1)^{i-1} i$ (see also [N, Section 9]).

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[4] Notice that $p_{134*}\iota_{ij*}(\gamma_1)$ is a degree $2(2n - i - j + 2)$ homology class of L and, therefore, must be zero.

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Department of Mathematics, MIT, 77 Massachusetts Avenue, Cambridge, MA 02139, USA
E-mail address: sasha_ts@mit.edu